RIESZ-TYPE INEQUALITIES AND OVERDETERMINED PROBLEMS FOR TRIANGLES AND QUADRILATERALS

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Abstract. We consider Riesz-type nonlocal interaction energies over convex polygons. We prove the analog of the Riesz inequality in this discrete setting for triangles and quadrilaterals, and obtain that among all N-gons with fixed area, the nonlocal energy is maximized by a regular polygon, for $N = 3, 4$. Further we derive necessary first-order stationarity conditions for a polygon with respect to a restricted class of variations, which will then be used to characterize regular N-gons, for $N = 3, 4$, as solutions to an overdetermined free boundary problem.

1. Introduction

In this paper we study a class of nonlocal repulsive energies of generalized Riesz-type on polygons. We consider the nonlocal energy

$$\mathcal{E}(E) := \int_E \int_E K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \tag{1.1}$$

defined on measurable subsets $E \subset \mathbb{R}^2$ with finite Lebesgue measure. We assume that the kernel K satisfies the following assumptions:

(K1) $K \in C^1((0,\infty))$, $K \geqslant 0$;

(K2) K is strictly decreasing;

(K3) K satisfies

$$\int_0^1 K(r)\,r\,\mathrm{d}r < \infty. \tag{1.2}$$

The kernel K is possibly singular at the origin, and the integrability condition (1.2) guarantees that the energy (1.1) is finite on sets with finite measure (see Remark 2.4). The prototype case is the Riesz kernel $K(r) = r^{-\alpha}$, with $\alpha \in (0,2)$.

It is well-known that the energy (1.1) (in any dimension) is uniquely maximized by the ball under volume constraint, as a consequence of Riesz's rearrangement inequality. Moreover, at least in the case of the Riesz kernels, balls are characterized as the unique critical points for the energy (1.1) under volume constraint, in the following sense. We define the *potential* associated to a measurable set $E \subset \mathbb{R}^2$ with finite measure as

$$v_E(x) := \int_E K(|x-y|)\,\mathrm{d}y, \tag{1.3}$$

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and say that a set E is stationary for $\mathcal{E}$ with respect to area-preserving variations if v_E is constant on ∂E. It was proved in a series of contributions [8, 9, 15, 18] via moving plane methods, and in full generality for Riesz kernels in [12] via a continuous Steiner symmerization argument, that balls are the only sets which enjoy this property: in other words, when defined over *all* measurable sets of fixed measure, the overdetermined problem for the potential enforces the symmetry of the set.

The scope of this paper is to investigate the same two questions in a discrete setting, namely when restricting the class of sets on which we evaluate the energy to convex polygons with a fixed number of sides. While on the one hand this restriction simplifies some aspects of the problem by essentially reducing it to a finite dimensional problem, on the other hand it introduces new challenges and requires new techniques, as classical arguments such as moving plane methods do not apply in restricted classes.

We first consider the problem of the area-constrained maximization of the nonlocal energy $\mathcal{E}$ in the class $\mathscr{P}_N$ of convex polygons in $\mathbb{R}^2$ with $N \geqslant 3$ sides: for $m > 0$,

$$\max\bigl\{\mathcal{E}(\mathcal{P}) \,:\, \mathcal{P} \in \mathscr{P}_N,\, |\mathcal{P}| = m\bigr\}, \tag{1.4}$$

where $|\mathcal{P}| \coloneqq \mathcal{L}^2(\mathcal{P})$ denotes the area of a polygon $\mathcal{P} \in \mathscr{P}_N$. It is in general expected that for each fixed number of sides the regular N-gon is the unique maximizer of (1.4). In our first main result we show that this is true in the case of triangles and quadrilaterals.

Theorem 1.1. *The equilater triangle is the unique (up to rigid movements) maximizer of $\mathcal{E}$ in $\mathscr{P}_3$ under area constraint, and the square is the unique (up to rigid movements) maximizer of $\mathcal{E}$ in $\mathscr{P}_4$ under area constraint.*

The proof relies on the combination of two properties that had already been established in the literature and are well-known to experts: (a) the fact that the nonlocal energy is increasing under Steiner symmetrization of a set; (b) the observation, originally due to Pólya and Szegő, that for any given triangle or quadrilateral it is possible to find a sequence of Steiner symmetrizations which converge to an equilateral triangle or to a square, respectively. This strategy was used by Pólya and Szegő [17, p. 158] to prove their conjecture about the optimality of the regular N-gon for various classical shape functionals, such as the principal eigenvalue of the Laplacian, the torsional rigidity, and the electrostatic capacity, for $N = 3, 4$. The main drawback of this approach is that, for more than four sides, it seems not possible to construct in an easy way a sequence of symmetrizations converging to the regular N-gon and preserving the number of sides at each step. Therefore the extension of Theorem 1.1 to the case $N \geqslant 5$ seems to be, as far as we know, an interesting open problem.

Besides the above mentioned conjecture by Pólya and Szegő, solved only for the logarithmic capacity in [19], the problem of optimality of regular N-gons for variational functionals has been the object of several contributions. Among these, we mention the papers [5, 10, 16], dealing with various shape optimization problems on polygons involving spectral functionals, and [4], where it is proved that the regular polygon minimizes the Cheeger constant among polygons with fixed area and number of sides.

Next, we turn to the second main question that we address in this paper, namely whether the regular N-gon is characterized by the stationarity conditions for problem (1.4), as it is the case for the ball. Of course, we need to consider a notion of criticality with respect to variations that preserve the polygonal structure and the number of sides. Following [11], in Section 3 we introduce two specific classes of perturbations of a given polygon: the first is obtained by translating a side of the polygon parallel to itself, the second by rotating a side with respect to its midpoint. We then show that, for $N = 3$ and $N = 4$ sides, the unique N-gon which is stationary with respect to these two families of perturbations is the equilateral triangle or the square, respectively.

In order to state precisely our second main result, we need to fix some notation that will be used throughout the paper. Given two points $P, Q \in \mathbb{R}^2$, we denote by $\overline{PQ} \coloneqq \{tP+(1-t)Q : t \in [0,1]\}$ the segment joining P and Q. For $N \geqslant 3$, let $\mathcal{P} \in \mathscr{P}_N$ be a polygon with N vertices $P_1, \ldots, P_N$. For notational convenience we also set $P_0 \coloneqq P_N$, $P_{N+1} \coloneqq P_1$. We let for $i \in \{1, \ldots, N\}$:

- ν_i be the exterior unit normal to the side $\overline{P_iP_{i+1}}$,
- ℓ_i be the length of the side $\overline{P_iP_{i+1}}$,
- θ_i be the (interior) angle at the vertex P_i,
- M_i be the midpoint of the side $\overline{P_iP_{i+1}}$.

Denoting by $v_{\mathcal{P}}$ the potential associated with the polygon $\mathcal{P}$ according to (1.3), we then consider the following two conditions:

$$\frac{1}{\ell_i}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) = \frac{1}{\ell_j}\int_{\overline{P_jP_{j+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) \quad \text{for all } i,j \in \{1,\ldots,N\}, \tag{1.5}$$

which corresponds to the criticality condition for the energy $\mathcal{E}$ under an area constraint, when sides are translated parallel to themselves, and

$$\int_{\overline{P_iM_i}} v_{\mathcal{P}}(x)|x-M_i|\,\mathrm{d}\mathcal{H}^1(x) = \int_{\overline{P_{i+1}M_i}} v_{\mathcal{P}}(x)|x-M_i|\,\mathrm{d}\mathcal{H}^1(x) \quad \text{for all } i \in \{1,\ldots,N\}, \tag{1.6}$$

which corresponds to the criticality condition for the energy $\mathcal{E}$ under an area constraint, when a side is rotated around its midpoint. The derivation of (1.5) and (1.6) will be given in Section 3, see in particular Theorem 3.7.

Our second result is the following.

Theorem 1.2. *If $\mathcal{P} \in \mathscr{P}_3$ obeys condition (1.6), then $\mathcal{P}$ is an equilateral triangle. If $\mathcal{P} \in \mathscr{P}_4$ obeys conditions (1.5) and (1.6), then $\mathcal{P}$ is a square.*

This result can be interpreted as a Serrin-type theorem yielding the characterization of the regular N-gon as the unique solution of the overdetermined problem (1.3)–(1.5)–(1.6) for the potential $v_{\mathcal{P}}$. Despite the large literature on overdetermined boundary value problems, symmetry results of this kind in a polygonal setting seem to have been considered only recently, with a first contribution by Fragalà and Velichkov [11] which was also inspirational for our work. In [11] it was proved that the overdetermined problem corresponding to the stationarity conditions for the torsional rigidity and for the first Dirichlet eigenvalue of the Laplacian, under an area or a perimeter constraint, characterizes the equilateral triangle among all triangles.

The proof of Theorem 1.2 in the case of triangles (see Section 4) is relatively simple and is based on a straightforward reflection argument. However, we also give a second proof which will be extended to the case of quadrilaterals in Section 5 (and, hopefully, might work in general for an arbitrary number of sides). This second argument is inspired by an idea of Carrillo et al. [7] and is based on a continuous symmetrization (in the spirit of the continuous Steiner symmetrization [3]), see also Figure 3. We show that, if two sides of a triangle have different lengths, then by translating the common vertex parallel to the third side the first variation of the energy is different from zero. In turn, since the criticality condition with respect to this variation can be expressed in terms of the conditions (1.5) and (1.6), we obtain that all sides of a critical triangle have to be equal.

The proof for quadrilaterals exploits the same idea, and uses a continuous symmetrization to prove that the conditions (1.5) and (1.6) enforce the property of being equilateral, thus reducing the proof to the class of rhombi; then in a second step we prove that the polygon has to be also equiangular, using a reflection argument.

We conjecture that Theorem 1.2 should be true for every fixed number $N \geqslant 3$ of sides, and that a possible strategy for the proof could follow the same ideas sketched above: one should first prove that the polygon is equilateral via continuous symmetrization, and then that it is equiangular via reflection. This strategy is somehow reminiscent of Zenodorus' classical proof of the isoperimetric property of the regular polygons [13]. Notice that a positive answer to this question would also provide an extension of Theorem 1.1 to the case $N \geqslant 5$. However, the study of the sign of the first variation in the case $N \geqslant 5$ is significantly more involved and seems to require new ideas. This will be the object of future work.

We also remark that, for $N = 3$, every triangle satisfies the conditions (1.5), which therefore do not yield symmetry at all (see Remark 4.1). However, in the case of quadrilaterals both (1.5) and (1.6) are required to characterize the square: indeed there exists quadrilaterals different from the square satisfying (1.5) but not (1.6) (e.g. rhombi), and quadrilaterals different from the square satisfying (1.6) but not (1.5) (e.g. rectangles).

We conclude this introduction by mentioning that our motivation for the study of this problem comes from our recent work [2] on an anisotropic nonlocal isoperimetric problem, recently introduced in [8] as an extension of the classical liquid drop model of Gamow, in which we considered the volume-constrained minimization of the sum of the nonlocal energy $\mathcal{E}$ and a crystalline anisotropic perimeter. Due to the presence of a surface tension whose Wulff shape (i.e. the corresponding isoperimetric region) is a convex polygon, it was shown that at least in the small mass regime minimizers of the total energy have a polygonal structure; this naturally led us to the question of characterizing the polygons which are stationary for the nonlocal energy $\mathcal{E}$.

Structure of the paper. The proof of Theorem 1.1 is given in Section 2 via Steiner symmetrization. In Section 3 we derive the identities (1.5) and (1.6) as stationarity conditions for the nonlocal energy with respect to two particular classes of variations. Finally, Section 4 and Section 5 contain the proof of Theorem 1.2 in the case $N = 3$ and $N = 4$, respectively.

2. Maximality of equilateral triangles and squares by Steiner symmetrization

In this section we will give a proof to Theorem 1.1. Our proof is based on Steiner symmetrization and a simple argument by Pólya and Szegő which describes two sequences of symmetrizations transforming a given triangle into an equilateral triangle and a given quadrilateral into a square, respectively.

We start by giving the necessary definitions and prove two lemmas regarding the role of Steiner symmetrization on the nonlocal energy $\mathcal{E}$: in particular, we show that the nonlocal energy is strictly increasing with respect to Steiner symmetrization of a set, unless the set is already symmetric. Since this property is not restricted to dimension 2, in the first part of this section we work in general dimension $d \geqslant 2$, and we replace assumption (K3) on the kernel by its general version

$$\int_0^1 K(r) r^{d-1}\,\mathrm{d}r < \infty. \tag{2.1}$$

The proof is essentially contained in [14, Chapter 3], but we include the details here to point out the properties that we need. See also [6] for details on rearrangement inequalities.

In the following, we denote by $e_1, \dots, e_d$ the vectors of the canonical basis of $\mathbb{R}^d$. We also denote the generic point of $\mathbb{R}^d \equiv \mathbb{R}^{d-1} \times \mathbb{R}$ by $x = (x', x_d)$.

Definition 2.1. Given any measurable set $E \subset \mathbb{R}^d$, its *symmetric rearrangement* is defined as $E^* := B_r$ with $\omega_d r^d = |E|$, where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Definition 2.2. For $E \subset \mathbb{R}^d$ and $x' \in \mathbb{R}^{d-1}$, let $E_{x'} := \big\{x_d \in \mathbb{R} \colon (x', x_d) \in E\big\}$. The *Steiner symmetrization* of E in the direction e_d is defined as

$$E^s := \Big\{(x', x_d) \in \mathbb{R}^d \colon x' \in \mathbb{R}^{d-1},\, x_d \in (E_{x'})^*\Big\}.$$

Notice that the Steiner symmetrization is a volume preserving operation.

The first lemma shows that Steiner symmetrization of a set E increases its nonlocal energy $\mathcal{E}$, and it follows from Riesz's rearrangement inequality in one dimension (see [14, Lemma 3.6]) and Fubini's theorem.

Lemma 2.3. *Let $E \subset \mathbb{R}^d$ be a measurable set with finite measure. Then*

$$\mathcal{E}(E) \leqslant \mathcal{E}(E^s).$$

Proof. We first prove the following property: given three measurable sets F, G, and $H \subset \mathbb{R}^d$ with finite measure, we have

$$\mathcal{I}(F, G, H) \leqslant \mathcal{I}(F^s, G^s, H^s), \tag{2.2}$$

where $\mathcal{I}(F, G, H) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_F(x)\chi_G(x-y)\chi_H(y)\,\mathrm{d}x\,\mathrm{d}y$. Indeed, by Fubini's theorem

$$\begin{aligned}
\mathcal{I}(F, G, H) &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_F(x', x_d)\chi_G(x'-y', x_d-y_d)\chi_H(y', y_d)\,\mathrm{d}x_d\,\mathrm{d}y_d\,\mathrm{d}x'\,\mathrm{d}y' \\
&= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{F_{x'}}(x_d)\chi_{G_{x'-y'}}(x_d-y_d)\chi_{H_{y'}}(y_d)\,\mathrm{d}x_d\,\mathrm{d}y_d\,\mathrm{d}x'\,\mathrm{d}y' \\
&\leqslant \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{(F_{x'})^*}(x_d)\chi_{(G_{x'-y'})^*}(x_d-y_d)\chi_{(H_{y'})^*}(y_d)\,\mathrm{d}x_d\,\mathrm{d}y_d\,\mathrm{d}x'\,\mathrm{d}y' \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{F^s}(x)\chi_{G^s}(x-y)\chi_{H^s}(y)\,\mathrm{d}x\,\mathrm{d}y = \mathcal{I}(F^s, G^s, H^s),
\end{aligned}$$

where the inequality follows from the one dimensional Riesz's rearrangement inequality.

Now, since the kernel K is strictly decreasing, for any $t > 0$ there exists $r(t) > 0$ such that $\{x \in \mathbb{R}^d \colon K(|x|) > t\} = B_{r(t)}$. Using the layer cake formula (see [14, Theorem 1.13]) and Fubini's theorem, we can rewrite the nonlocal energy as

$$\begin{aligned}
\mathcal{E}(E) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_E(x) K(|x-y|)\chi_E(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_E(x)\left(\int_0^\infty \chi_{\{K>t\}}(|x-y|)\,\mathrm{d}t\right)\chi_E(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_E(x)\chi_{B_{r(t)}}(x-y)\chi_E(y)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}t.
\end{aligned}$$

Then (2.2) implies that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_E(x)\chi_{B_{r(t)}}(x-y)\chi_E(y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{E^s}(x)\chi_{B_{r(t)}}(x-y)\chi_{E^s}(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Hence, rewriting the energy of E^s using Fubini's theorem and the layer cake representation as above, we get $\mathcal{E}(E) \leqslant \mathcal{E}(E^s)$. □

Remark 2.4. Notice that for every measurable set $E \subset \mathbb{R}^d$ with finite measure, in view of the assumption (2.1) and of the monotonicity of K, the potential v_E defined in (1.3) is a bounded function:

$$\begin{aligned} v_E(x) &= \int_{E\cap B_1(x)} K(|x-y|)\,\mathrm{d}y + \int_{E\setminus B_1(x)} K(|x-y|)\,\mathrm{d}y \\ &\leqslant \int_{B_1} K(|y|)\,\mathrm{d}y + K(1)|E\setminus B_1(x)| \\ &\leqslant d\int_0^1 K(r)r^{d-1}\,\mathrm{d}r + K(1)|E| =: C(d,K,|E|) < \infty. \end{aligned}$$

In turn, the energy of E is finite: $\mathcal{E}(E) = \int_E v_E(x)\,\mathrm{d}x \leqslant C(d,K,|E|)|E|$.

The next lemma shows that if a set and its Steiner symmetral have the same nonlocal energy, then they are translates of each other almost everywhere.

Lemma 2.5. *Let $E \subset \mathbb{R}^d$ be a measurable set with finite measure. Then $\mathcal{E}(E) = \mathcal{E}(E^s)$ only if $|E \triangle (E^s + y_0)| = 0$ for some $y_0 \in \mathbb{R}^d$, where $\triangle$ denotes the symmetric difference of two sets.*

Proof. For dimension $d = 1$, the result follows from [14, Theorem 3.9]. Let $\kappa(x) := K(|x|)$ for $x \in \mathbb{R}^d$. For $d > 1$, we note that, by Fubini's theorem, the equality $\mathcal{E}(E) = \mathcal{E}(E^s)$ is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_E(x',x_d)\chi_E(y',y_d)\,\kappa(x'-y',x_d-y_d)\,\mathrm{d}x_d\,\mathrm{d}y_d\,\mathrm{d}x'\,\mathrm{d}y' \\ &\qquad = \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{E^s}(x',x_d)\chi_{E^s}(y',y_d)\,\kappa(x'-y',x_d-y_d)\,\mathrm{d}x_d\,\mathrm{d}y_d\,\mathrm{d}x'\,\mathrm{d}y'. \end{aligned}$$

Since $\chi_{E^s}(x',x_d) = \chi_{(E_{x'})^*}(x_d)$, defining

$$\mathcal{I}^1\big(E_{x'},\kappa(x'-y',\,\cdot\,),E_{y'}\big) := \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{E_{x'}}(x_d)\chi_{E_{y'}}(y_d)\,\kappa(x'-y',x_d-y_d)\,\mathrm{d}x_d\,\mathrm{d}y_d$$

the above equation becomes

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \mathcal{I}^1\big(E_{x'},\kappa(x'-y',\,\cdot\,),E_{y'}\big)\,\mathrm{d}x'\,\mathrm{d}y' \\ &\qquad = \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \mathcal{I}^1\big((E_{x'})^*,\kappa(x'-y',\,\cdot\,),(E_{y'})^*\big)\,\mathrm{d}x'\,\mathrm{d}y'. \end{aligned}$$

In turn, since by Riesz's rearrangement inequality (cf. [14, Theorem 3.7])

$$\mathcal{I}^1\big(E_{x'},\kappa(x'-y',\,\cdot\,),E_{y'}\big) \leqslant \mathcal{I}^1\big((E_{x'})^*,\kappa(x'-y',\,\cdot\,),(E_{y'})^*\big),$$

we get that

$$\mathcal{I}^1\big(E_{x'},\kappa(x'-y',\,\cdot\,),E_{y'}\big) = \mathcal{I}^1\big((E_{x'})^*,\kappa(x'-y',\,\cdot\,),(E_{y'})^*\big),$$

for a.e. x', $y' \in \mathbb{R}^{d-1}$. This implies, by the one dimensional result, that $E_{x'}$ and $E_{y'}$ are both intervals centered at the same point for a.e. $(x', y') \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$. Moreover, this point is independent of (x', y') as we can repeat the argument for any $(x', \tilde{y}')$ with $\tilde{y}' \in \mathbb{R}^{d-1}$ and obtain that the centers of $E_{x'}$ and $E_{\tilde{y}'}$ coincide. Therefore the set E, after possibly a translation in the x_d direction, is Steiner symmetric up to a set of measure zero, i.e., $|E \triangle (E^s + y_0)| = 0$ for some $y_0 \in \mathbb{R}^d$. $\square$

We are now ready to prove our first main result which relies on an argument by Pólya and Szegő that we detail here.

Proof of Theorem 1.1. Let $\mathcal{P}_0 \in \mathscr{P}_3$ be an arbitrary triangle with $|\mathcal{P}_0| = 1$. Following [17, Section 7.4], we will describe an infinite sequence of Steiner symmetrizations of $\mathcal{P}_0$ which will transform it into an equilateral triangle. To this end, let $2a_0$ be the length of one of the sides of $\mathcal{P}_0$. Then the corresponding altitude perpendicular to this side has length a_0^{-1}. By Steiner symmetrization of $\mathcal{P}_0$ in the direction of this side, we obtain an isosceles triangle $\mathcal{P}_1$ where the length of equal sides is $a_1 = (a_0^2 + a_0^{-2})^{1/2}$. Next we symmetrize $\mathcal{P}_1$ in the direction of one of the equal sides to obtain another isosceles triangle $\mathcal{P}_2$ with equal sides of length $a_2 = (a_1^2/4 + 4/a_1^2)^{1/2}$. Repeating this process, we see that the length of the equal sides of the isosceles triangle $\mathcal{P}_n$ is given recursively by $a_n = (a_{n-1}^2/4 + 4/a_{n-1}^2)^{1/2}$ with $n \geqslant 2$. Taking the limit $n \to \infty$ we see that $a_n \to 2/\sqrt[4]{3}$, and since in each iteration the area of $\mathcal{P}_n$ is one, in the limit, we obtain that all three sides are of length $2/\sqrt[4]{3}$.

Now, suppose $\mathcal{P}_0 \in \mathscr{P}_4$ is an arbitrary quadrilateral with $|\mathcal{P}_0| = 1$. Symmetrizing $\mathcal{P}_0$ in the direction of one of its diagonals we obtain a kite, $\mathcal{P}_1$ (that is, a quadrilateral with a diagonal as axis of symmetry). Next, we symmetrize $\mathcal{P}_1$ in the direction of its axis of symmetry and obtain a rhombus, $\mathcal{P}_2$. Let a_2 be the side length of $\mathcal{P}_2$. Symmetrizing $\mathcal{P}_2$ in the direction of one of its sides we get a rectangle $\mathcal{P}_3$ such that its longer side has length $a_3 = a_2$. Symmetrizing $\mathcal{P}_3$ in the direction of one of its diagonals we obtain another rhombus, $\mathcal{P}_4$, with side length $a_4 = \big(a_3^2/(a_3^4+1) + (a_3^4+1)/(4a_3^2)\big)^{1/2}$. Continuing this process we will obtain a sequence of quadrilaterals such that $\mathcal{P}_n$ is a rhombus for n even, and a rectangle for n odd. If a_n denotes the side length of $\mathcal{P}_n$ (n even) or the length of the longer side (n odd), we have by construction

$$a_{2n+1} = a_{2n}, \qquad a_{2n} = \sqrt{\frac{a_{2n-1}^2}{a_{2n-1}^4+1} + \frac{a_{2n-1}^4+1}{4a_{2n-1}^2}} \qquad \text{for } n \geqslant 2,$$

recursively. Taking the limit $n \to \infty$ we get that $a_n \to 1$; hence, in the limit successive symmerizations of $\mathcal{P}_0$ yield a square.

Since, by Lemma 2.3, Steiner symmetrization increases the nonlocal energy $\mathcal{E}$, we obtain that among the classes $\mathscr{P}_3$ and $\mathscr{P}_4$ an equilateral triangle and a square maximize $\mathcal{E}$, respectively. The uniqueness of the maximizer in each class, up to rigid movements, follows from Lemma 2.5. $\square$

3. Stationarity conditions: sliding and tilting

We derive the stationarity conditions for the nonlocal energy (1.1) under an area constraint, with respect to two particular classes of perturbations of a polygon $\mathcal{P} \in \mathscr{P}_N$, obtained by sliding one side parallel to itself, or tilting one side around its midpoint. More precisely, we consider the following two families of one-parameter deformations. In the following, we assume that $\mathcal{P} \in \mathscr{P}_N$ is a given polygon with $N \geqslant 3$ vertices $P_1, \ldots, P_N$.

Definition 3.1 (Sliding of one side)**.** Fix a side $\overline{P_iP_{i+1}}$, $i \in \{1,\dots,N\}$. For $t \in \mathbb{R}$ with $|t|$ sufficiently small, we define the polygon $\mathcal{P}_t \in \mathscr{P}_N$ with vertices $P_1^t,\dots,P_N^t$ obtained as follows (see Figure 1):

(i) all vertices except P_i and P_{i+1} are fixed, i.e. $P_j^t := P_j$ for all $j \in \{1,\dots N\}\setminus\{i,i+1\}$;

(ii) the vertices P_i^t and P_{i+1}^t lie on the lines containing $\overline{P_{i-1}P_i}$ and $\overline{P_{i+1}P_{i+2}}$, respectively;

(iii) the side $\overline{P_i^tP_{i+1}^t}$ is parallel to $\overline{P_iP_{i+1}}$ and at a distance $|t|$ from $\overline{P_iP_{i+1}}$, in the direction of ν_i if $t>0$ and in the direction of $-\nu_i$ if $t<0$.

Explicitly:

$$P_i^t := P_i + \frac{t}{\sin\theta_i}\frac{P_i - P_{i-1}}{|P_i - P_{i-1}|}, \qquad P_{i+1}^t := P_{i+1} + \frac{t}{\sin\theta_{i+1}}\frac{P_{i+1} - P_{i+2}}{|P_{i+1} - P_{i+2}|}.$$

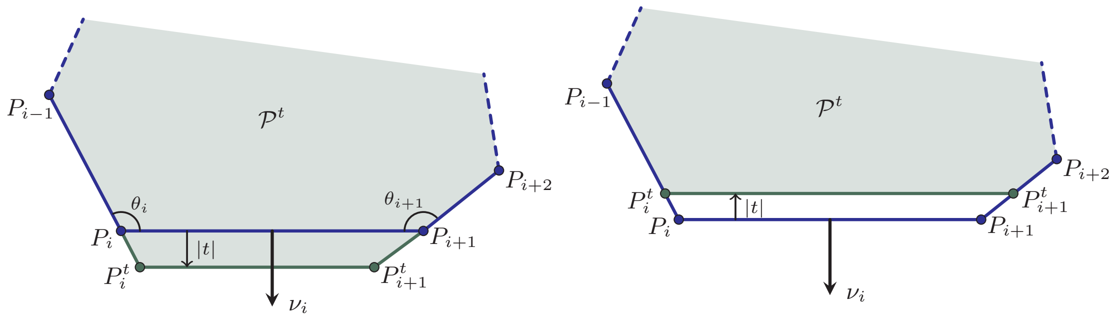

FIGURE 1. A polygon $\mathcal{P}$ and its variation $\mathcal{P}^t$ (shaded region) as in Definition 3.1, obtained by sliding the side $\overline{P_iP_{i+1}}$ in the normal direction at a distance $|t|$: the case $t>0$ (left) and $t<0$ (right).

Definition 3.2 (Tilting of one side)**.** Fix a side $\overline{P_iP_{i+1}}$, $i \in \{1,\dots,N\}$. For $t \in \mathbb{R}$ with $|t|$ sufficiently small, we define the polygon $\mathcal{P}_t \in \mathscr{P}_N$ with vertices $P_1^t,\dots,P_N^t$ obtained as follows (see Figure 2):

(i) all vertices except P_i and P_{i+1} are fixed, i.e. $P_j^t := P_j$ for all $j \in \{1,\dots N\}\setminus\{i,i+1\}$;

(ii) the vertices P_i^t and P_{i+1}^t lie on the lines containing $\overline{P_{i-1}P_i}$ and $\overline{P_{i+1}P_{i+2}}$, respectively;

(iii) the line containing $\overline{P_i^tP_{i+1}^t}$ is obtained by rotating the line containing $\overline{P_iP_{i+1}}$ around the midpoint M_i of $\overline{P_iP_{i+1}}$ by an angle t;

(iv) the direction of rotation is such that, for $t>0$, the point P_{i+1}^t belongs to the segment $\overline{P_{i+1}P_{i+2}}$, while for $t<0$ the point P_i^t belongs to the segment $\overline{P_{i-1}P_i}$.

Explicitly:

$$P_i^t := P_i + \frac{\ell_i \sin t}{2\sin(\theta_i - t)}\frac{P_i - P_{i-1}}{|P_i - P_{i-1}|}, \qquad P_{i+1}^t := P_{i+1} - \frac{\ell_i \sin t}{2\sin(\theta_{i+1} + t)}\frac{P_{i+1} - P_{i+2}}{|P_{i+1} - P_{i+2}|}.$$

In Proposition 3.4 and Proposition 3.5 below we compute the first variation of the nonlocal energy (1.1) and of the area of a polygon $\mathcal{P}$ with respect to these two classes of perturbations. Before doing that, we prove a first variation formula for the nonlocal energy with respect to a general perturbation. The derivation is valid also in any dimension $d \geqslant 2$, replacing the assumption (K3) by (2.1).

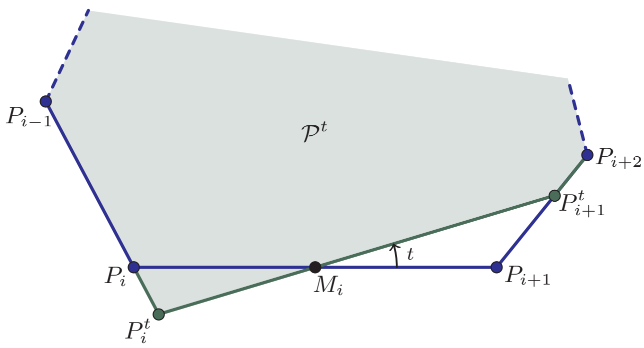

FIGURE 2. A polygon $\mathcal{P}$ and its variation $\mathcal{P}^t$ (shaded region) as in Definition 3.2, obtained by tilting the side $\overline{P_iP_{i+1}}$ around its midpoint M_i by an angle $t > 0$.

Proposition 3.3 (First variation of $\mathcal{E}$). *Let $E \subset \mathbb{R}^2$ be a bounded open set with piecewise smooth boundary. Let $\Phi : \mathbb{R}^2 \times [-\bar{t}, \bar{t}] \to \mathbb{R}^2$, for $\bar{t} > 0$, be a flow of class C^2 such that $\Phi(x, 0) = x$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{E}(\Phi_t(E)) = 2 \int_{\partial E} v_E(x)\, X(x) \cdot \nu_E(x)\, \mathrm{d}\mathcal{H}^1(x), \tag{3.1}$$

where $X(x) := \frac{\partial \Phi(x,t)}{\partial t}|_{t=0}$ is the initial velocity, v_E is the potential of E defined in (1.3), and ν_E is the exterior unit normal on ∂E.

Proof. The proof follows the same strategy used in [1] to compute the first variation in the particular case of a Riesz kernel. We regularize the kernel by introducing a small parameter $\delta > 0$ and by setting

$$K_\delta(r) := K(r + \delta), \qquad \mathcal{E}_\delta(E) := \int_E \int_E K_\delta(|x - y|)\, \mathrm{d}x\, \mathrm{d}y, \tag{3.2}$$

so that $K_\delta \in C^1([0, +\infty))$. By using the kernel K_δ we can bring the derivative inside the integral and all the following computations are justified.

We let $\Phi_t(x) := \Phi(x, t)$ and $J\Phi_t(x) := \det(D\Phi_t(x))$ denote the Jacobian of the map Φ_t. By a change of variables we obtain for $t \in (-\bar{t}, \bar{t})$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta(\Phi_t(E)) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_E \int_E K_\delta(|\Phi_t(x) - \Phi_t(y)|) J\Phi_t(x) J\Phi_t(y)\, \mathrm{d}x\, \mathrm{d}y \\
&= 2 \int_E \int_E K_\delta(|\Phi_t(x) - \Phi_t(y)|) J\Phi_t(x) \frac{\partial J\Phi_t}{\partial t}(y)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad + 2 \int_E \int_E K_\delta'(|\Phi_t(x) - \Phi_t(y)|) \frac{\Phi_t(x) - \Phi_t(y)}{|\Phi_t(x) - \Phi_t(t)|} \cdot \frac{\partial \Phi(x,t)}{\partial t}\, J\Phi_t(x) J\Phi_t(y)\, \mathrm{d}x\, \mathrm{d}y \\
&= 2 \int_E \int_E K_\delta(|\Phi_t(x) - \Phi_t(y)|) J\Phi_t(x) \frac{\partial J\Phi_t}{\partial t}(y)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad + 2 \int_E \int_E \Big[\nabla_x \big(K_\delta(|\Phi_t(x) - \Phi_t(y)|)\big) \big(D\Phi_t(x)\big)^{-1}\Big] \cdot \frac{\partial \Phi(x,t)}{\partial t}\, J\Phi_t(x) J\Phi_t(y)\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

Now integrating by parts in the last integral we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta(\Phi_t(E)) &= 2\int_E\int_E K_\delta(|\Phi_t(x)-\Phi_t(y)|)J\Phi_t(x)\frac{\partial J\Phi_t}{\partial t}(y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad - 2\int_E\int_E K_\delta(|\Phi_t(x)-\Phi_t(y)|)\operatorname{div}_x\Big(\frac{\partial\Phi(x,t)}{\partial t}\big(D\Phi_t(x)\big)^{-T}J\Phi_t(x)J\Phi_t(y)\Big)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + 2\int_E\int_{\partial E} K_\delta(|\Phi_t(x)-\Phi_t(y)|)\frac{\partial\Phi(x,t)}{\partial t}\big(D\Phi_t(x)\big)^{-T}\cdot\nu_E(x)J\Phi_t(x)J\Phi_t(y)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}y,\end{aligned}$$

that is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta(\Phi_t(E)) &= \int_E\int_E K_\delta(|\Phi_t(x)-\Phi_t(y)|)h_1(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \int_E\int_{\partial E} K_\delta(|\Phi_t(x)-\Phi_t(y)|)h_2(x,y,t)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}y,\end{aligned}$$

where we set

$$\begin{aligned}h_1(x,y,t) &\coloneqq 2J\Phi_t(x)\frac{\partial J\Phi_t}{\partial t}(y) - 2\operatorname{div}_x\Big(\frac{\partial\Phi(x,t)}{\partial t}\big(D\Phi_t(x)\big)^{-T}J\Phi_t(x)J\Phi_t(y)\Big),\\ h_2(x,y,t) &\coloneqq 2\frac{\partial\Phi(x,t)}{\partial t}\big(D\Phi_t(x)\big)^{-T}\cdot\nu_E(x)J\Phi_t(x)J\Phi_t(y).\end{aligned}$$

By using the definition (3.2) of K_δ and the fact that the functions $h_1(x,y,t)$ and $h_2(x,y,t)$ are uniformly bounded, one can then show that $\mathcal{E}_\delta(\Phi_t(E))\to\mathcal{E}(\Phi_t(E))$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta(\Phi_t(E)) \to H(t) &\coloneqq \int_E\int_E K(|\Phi_t(x)-\Phi_t(y)|)h_1(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \int_E\int_{\partial E} K(|\Phi_t(x)-\Phi_t(y)|)h_2(x,y,t)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}y,\end{aligned}$$

as $\delta\to0$, uniformly with respect to $t\in[-\bar{t},\bar{t}]$. Therefore we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\Phi_t(E)) = H(0) = 2\int_E\int_{\partial E}K(|x-y|)X(x)\cdot\nu_E(x)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}y,$$

where we used the Taylor expansion $\Phi_t(x)=x+tX(x)+o(t)$, from which it follows, in particular, the identity $\frac{\partial J\Phi_t}{\partial t}|_{t=0}=\operatorname{div}X$. □

We can now use the first variation formula (3.1) to compute the derivative of the energy along the perturbations of a polygon introduced in Definitions 3.1 and 3.2.

Proposition 3.4 (Sliding first variation)**.** *Let $\mathcal{P}\in\mathscr{P}_N$ and let $\{\mathcal{P}_t\}_t$ be the family of perturbations of $\mathcal{P}$ as in Definition 3.1, obtained by sliding the side $\overline{P_iP_{i+1}}$ parallel to itself. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\mathcal{P}_t) = 2\int_{\overline{P_iP_{i+1}}}v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x), \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}|\mathcal{P}_t| = \ell_i. \tag{3.4}$$

Proof. The flow $\{\Phi_t\}_t$ which induces the perturbation $\{\mathcal{P}_t\}_t$ obeys $(\Phi_t(x)-x)\cdot\nu_i=t$ for all $x\in\overline{P_iP_{i+1}}$; therefore its initial velocity has normal component

$$X\cdot\nu_i = 1 \qquad \text{on } \overline{P_iP_{i+1}}$$

and $X \cdot \nu_j = 0$ for all $j \neq i$. Hence (3.3) follows from Proposition 3.3. The first variation of the area (3.4) is computed in [11, Lemma 2.7] in the case of a triangle, but the proof is obviously the same for a general polygon, and follows from the identity

$$|\mathcal{P}_t| = |\mathcal{P}| + \ell_i t + o(t) \qquad \text{as } t \to 0,$$

which can be checked by elementary geometric arguments. □

Proposition 3.5 (Tilting first variation). *Let $\mathcal{P} \in \mathscr{P}_N$ and let $\{\mathcal{P}_t\}_t$ be the family of perturbations of $\mathcal{P}$ as in Definition 3.2, obtained by tilting the side $\overline{P_iP_{i+1}}$ with respect to its midpoint M_i. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\mathcal{P}_t) = 2\int_{\overline{P_iM_i}} v_{\mathcal{P}}(x)|x - M_i|\,\mathrm{d}\mathcal{H}^1(x) - 2\int_{\overline{M_iP_{i+1}}} v_{\mathcal{P}}(x)|x - M_i|\,\mathrm{d}\mathcal{H}^1(x), \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}|\mathcal{P}_t| = 0. \tag{3.6}$$

Proof. We can explicitly write a flow $\{\Phi_t\}_t$ which induces the perturbation $\{\mathcal{P}_t\}_t$: on the side $\overline{P_iP_{i+1}}$ it is given by

$$\Phi_t(x) = \begin{cases} x - \dfrac{\sin t}{\sin(\theta_i - t)}|x - M_i|\tau_i & \text{if } x \in \overline{P_iM_i}, \\[2ex] x + \dfrac{\sin t}{\sin(\theta_{i+1} + t)}|x - M_i|\tau_{i+1} & \text{if } x \in \overline{M_iP_{i+1}}, \end{cases}$$

where $\tau_i = \frac{1}{\ell_{i-1}}(P_{i-1} - P_i)$ and $\tau_{i+1} = \frac{1}{\ell_{i+1}}(P_{i+2} - P_{i+1})$ are the unit vectors parallel to the sides $\overline{P_{i-1}P_i}$ and $\overline{P_{i+1}P_{i+2}}$, respectively. Then the normal component of the initial velocity is

$$X(x) \cdot \nu_i = \begin{cases} -\dfrac{|x - M_i|}{\sin\theta_i}\,\tau_i \cdot \nu_i = |x - M_i| & \text{if } x \in \overline{P_iM_i}, \\[2ex] \dfrac{|x - M_i|}{\sin\theta_{i+1}}\,\tau_{i+1} \cdot \nu_i = -|x - M_i| & \text{if } x \in \overline{M_iP_{i+1}} \end{cases}$$

(and $X(x) \cdot \nu_j$ for $x \in \overline{P_jP_{j+1}}$, $j \neq i$). We obtain (3.5) by applying Proposition 3.3. (Notice that the flow $\{\Phi_t\}_t$ does not satisfy the regularity assumption in Proposition 3.3, but nevertheless it can be checked that the first variation formula (3.1) holds for this specific perturbation).

The first variation of the area (3.6) is computed in [11, Lemma 2.6] in the case of a triangle, but the proof is obviously the same for a general polygon, and follows from the identity

$$|\mathcal{P}_t| = |\mathcal{P}| + o(t) \qquad \text{as } t \to 0,$$

which can be checked by elementary geometric arguments. □

We are now ready to show that the equations (1.5) and (1.6) are the stationarity conditions for the nonlocal energy under an area constraint, with respect to the variations considered in Definitions 3.1 and 3.2 respectively.

Definition 3.6 (Stationarity). Let $\mathcal{P} \in \mathscr{P}_N$ and let $\{\mathcal{P}_t\}_t$ be a one-parameter deformation of $\mathcal{P}$, such as those considered before. We define an area-preserving variation by setting

$$\widetilde{\mathcal{P}}_t := \lambda_t\mathcal{P}_t \qquad \text{where } \lambda_t := \left(\frac{|\mathcal{P}|}{|\mathcal{P}_t|}\right)^{\frac{1}{2}}, \tag{3.7}$$

so that $|\widetilde{\mathcal{P}}_t| = |\mathcal{P}|$ for all t. We say that $\mathcal{P}$ is *stationary with respect to the variation* $\{\mathcal{P}_t\}_t$ if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\widetilde{\mathcal{P}}_t) = 0. \tag{3.8}$$

Theorem 3.7 (Stationarity conditions)**.** *A polygon $\mathcal{P} \in \mathscr{P}_N$ is stationary*

(i) *with respect to the family of perturbations in Definition 3.1 on the i-th side, for $i \in \{1, \dots, N\}$, if and only if*

$$\frac{1}{\ell_i}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) = \mu \tag{3.9}$$

where $\mu := \frac{1}{|\mathcal{P}|}\int_{\partial\mathcal{P}} v_{\mathcal{P}}(x)\,x\cdot\nu_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x)$ is a constant independent of i;

(ii) *with respect to the family of perturbations in Definition 3.2 on the i-th side, for $i \in \{1, \dots, N\}$, if and only if*

$$\int_{\overline{P_iM_i}} v_{\mathcal{P}}(x)|x - M_i|\,\mathrm{d}\mathcal{H}^1(x) = \int_{\overline{P_{i+1}M_i}} v_{\mathcal{P}}(x)|x - M_i|\,\mathrm{d}\mathcal{H}^1(x). \tag{3.10}$$

Proof. Let $\{\Phi_t\}_t$ be a flow such that $\mathcal{P}_t = \Phi_t(\mathcal{P})$ and let $X(x) := \frac{\partial\Phi(x,t)}{\partial t}|_{t=0}$ be the initial velocity. We compose the flow $\{\Phi_t\}_t$ with a rescaling which restores the area constraint: more precisely, we define

$$\Psi(x,t) := \lambda_t\Phi(x,t)$$

where λ_t is defined in (3.7). Notice that $\Psi_t(\mathcal{P}) = \lambda_t\Phi_t(\mathcal{P}) = \lambda_t\mathcal{P}_t = \widetilde{\mathcal{P}}_t$, and the initial velocity of the flow $\{\Psi_t\}_t$ is given by

$$Y(x) := \frac{\partial\Psi(x,t)}{\partial t}\Big|_{t=0} = X(x) + \frac{\mathrm{d}\lambda_t}{\mathrm{d}t}\Big|_{t=0} x = X(x) - \frac{1}{2|\mathcal{P}|}\frac{\mathrm{d}|\mathcal{P}_t|}{\mathrm{d}t}\Big|_{t=0} x.$$

Then by Proposition 3.3 we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\widetilde{\mathcal{P}}_t) &= 2\int_{\partial\mathcal{P}} v_{\mathcal{P}}(x)\,Y(x)\cdot\nu_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x)\\
&= 2\int_{\partial\mathcal{P}} v_{\mathcal{P}}(x)\,X(x)\cdot\nu_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) - \frac{1}{|\mathcal{P}|}\frac{\mathrm{d}|\mathcal{P}_t|}{\mathrm{d}t}\Big|_{t=0}\int_{\partial\mathcal{P}} v_{\mathcal{P}}(x)\,x\cdot\nu_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\mathcal{P}_t) - \frac{1}{|\mathcal{P}|}\frac{\mathrm{d}|\mathcal{P}_t|}{\mathrm{d}t}\Big|_{t=0}\int_{\partial\mathcal{P}} v_{\mathcal{P}}(x)\,x\cdot\nu_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x).
\end{aligned}$$

Therefore if $\mathcal{P}$ is stationary with respect to the perturbation $\{\mathcal{P}_t\}_t$, that is if (3.8) holds, then there exists a constant μ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\mathcal{P}_t) = \mu\,\frac{\mathrm{d}|\mathcal{P}_t|}{\mathrm{d}t}\Big|_{t=0}, \tag{3.11}$$

with μ explicitly given by

$$\mu := \frac{1}{|\mathcal{P}|}\int_{\partial\mathcal{P}} v_{\mathcal{P}}(x)\,x\cdot\nu_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x).$$

The conclusion follows by inserting in (3.11) the first variation formulas obtained in Propositions 3.4 and 3.5. $\square$

3.1. **Another family of volume-preserving variations.** In this subsection we introduce a third family of area-preserving perturbations of a polygon. We compute the corresponding first variation of the nonlocal energy and we show that it can be written as a combination of the sliding and tilting first variations.

Definition 3.8. Fix three consecutive vertices P_{i-1}, P_i, P_{i+1}, $i \in \{1, \dots, N\}$, of the polygon $\mathcal{P}$. For $t \in \mathbb{R}$ with $|t|$ sufficiently small, we define the polygon $\mathcal{P}_t \in \mathscr{P}_N$ with vertices $P_1^t, \dots, P_N^t$ obtained as follows (see Figure 3):

(i) all vertices except P_i are fixed, i.e. $P_j^t := P_j$ for all $j \in \{1, \dots N\} \setminus \{i\}$;

(ii) the vertex P_i^t is given by

$$P_i^t = P_i + t\,\frac{P_{i+1} - P_{i-1}}{|P_{i+1} - P_{i-1}|},$$

that is, P_i^t lies on the line through P_i parallel to the diagonal $\overline{P_{i-1}P_{i+1}}$, at a distance $|t|$ from P_i.

Notice that the previous perturbation is area preserving: $|\mathcal{P}_t| = |\mathcal{P}|$.

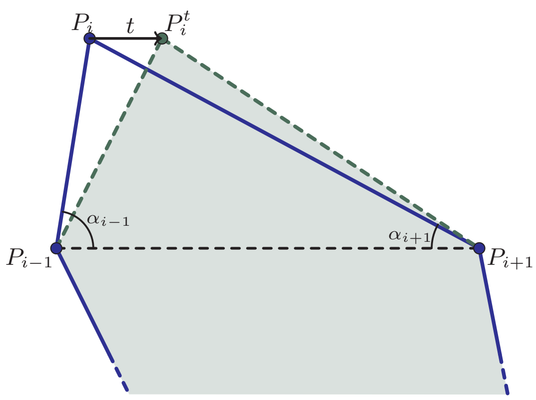

FIGURE 3. A polygon $\mathcal{P}$ and its variation $\mathcal{P}^t$ (shaded region) as in Definition 3.8, obtained by moving the vertex P_i parallel to the diagonal $\overline{P_{i-1}P_{i+1}}$ at a distance $t > 0$.

Proposition 3.9. *Let $\mathcal{P} \in \mathscr{P}_N$ and let $\{\mathcal{P}_t\}_t$ be the family of perturbations of $\mathcal{P}$ as in Definition 3.8, obtained by moving the vertex P_i parallel to the diagonal $\overline{P_{i-1}P_{i+1}}$. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\mathcal{P}_t) = {}& \frac{2\sin\alpha_{i+1}}{\ell_i}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)|x - P_{i+1}|\,\mathrm{d}\mathcal{H}^1(x) \\ & - \frac{2\sin\alpha_{i-1}}{\ell_{i-1}}\int_{\overline{P_{i-1}P_i}} v_{\mathcal{P}}(x)|x - P_{i-1}|\,\mathrm{d}\mathcal{H}^1(x),\end{aligned} \tag{3.12}$$

where $\alpha_{i-1} \in (0, \theta_{i-1}]$ is the angle between $\overline{P_{i-1}P_{i+1}}$ and $\overline{P_{i-1}P_i}$, and $\alpha_{i+1} \in (0, \theta_{i+1}]$ is the angle between $\overline{P_{i-1}P_{i+1}}$ and $\overline{P_iP_{i+1}}$.

Proof. A flow $\{\Phi_t\}_t$ which induces the perturbation $\{\mathcal{P}_t\}_t$ is explicitly given by

$$\Phi_t(x) = \begin{cases} x + t\,\dfrac{|x-P_{i-1}|}{\ell_{i-1}}\,\tau & \text{if } x \in \overline{P_{i-1}P_i}, \\ x + t\,\dfrac{|x-P_{i+1}|}{\ell_i}\,\tau & \text{if } x \in \overline{P_iP_{i+1}}, \end{cases} \qquad \text{where} \quad \tau := \frac{P_{i+1}-P_{i-1}}{|P_{i+1}-P_{i-1}|}. \tag{3.13}$$

Then the normal component of the initial velocity is

$$X\cdot\nu_{i-1} = -\frac{\sin\alpha_{i-1}}{\ell_{i-1}}|x-P_{i-1}| \qquad \text{on } \overline{P_{i-1}P_i},$$
$$X\cdot\nu_i = \frac{\sin\alpha_{i+1}}{\ell_i}|x-P_{i+1}| \qquad \text{on } \overline{P_iP_{i+1}},$$

and we obtain (3.12) by applying Proposition 3.3. □

Proposition 3.10. *Assume that the polygon $\mathcal{P}$ satisfies* (1.5) *and* (1.6)*. Then the quantity* (3.12) *is zero for every $i \in \{1,\dots,N\}$.*

Proof. Denote by I the right-hand side in (3.12). Then

$$\begin{aligned}
I &= \frac{2\sin\alpha_{i+1}}{\ell_i}\left[\int_{\overline{P_iM_i}} v_{\mathcal{P}}(x)\Big(|x-M_i| + \frac{\ell_i}{2}\Big) + \int_{\overline{M_iP_{i+1}}} v_{\mathcal{P}}(x)\Big(\frac{\ell_i}{2} - |x-M_i|\Big)\right] \\
&\quad - \frac{2\sin\alpha_{i-1}}{\ell_{i-1}}\left[\int_{\overline{P_{i-1}M_{i-1}}} v_{\mathcal{P}}(x)\Big(\frac{\ell_{i-1}}{2} - |x-M_{i-1}|\Big) + \int_{\overline{M_{i-1}P_i}} v_{\mathcal{P}}(x)\Big(|x-M_{i-1}| + \frac{\ell_{i-1}}{2}\Big)\right] \\
&\overset{(1.6)}{=} \frac{2\sin\alpha_{i+1}}{\ell_i}\left[\frac{\ell_i}{2}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x)\right] - \frac{2\sin\alpha_{i-1}}{\ell_{i-1}}\left[\frac{\ell_{i-1}}{2}\int_{\overline{P_{i-1}P_i}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x)\right] \\
&= \sin\alpha_{i+1}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) - \sin\alpha_{i-1}\int_{\overline{P_{i-1}P_i}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) \\
&= \ell_i\sin\alpha_{i+1}\left(\frac{1}{\ell_i}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) - \frac{1}{\ell_{i-1}}\int_{\overline{P_{i-1}P_i}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x)\right) \overset{(1.5)}{=} 0,
\end{aligned}$$

where we used the identity $\frac{\ell_{i-1}}{\sin\alpha_{i+1}} = \frac{\ell_i}{\sin\alpha_{i-1}}$. □

Notice that, if we move the vertex P_i according to Definition 3.8, to conclude that the corresponding first variation is zero it is sufficient that (1.5) and (1.6) hold for the two sides $\overline{P_{i-1}P_i}$ and $\overline{P_iP_{i+1}}$.

The strategy to prove Theorem 1.2 for quadrilaterals is mainly based on the previous proposition: indeed, we will prove that if two consecutive sides have different lengths $\ell_{i-1} \neq \ell_i$, then the first variation of the nonlocal energy with respect to the perturbation in Definition 3.8 is different from zero; in turn, by Proposition 3.10 the quadrilateral does not satisfy the stationarity conditions (1.5) and (1.6). As a consequence, a quadrilateral satisfying both (1.5) and (1.6) is necessarily equilateral (that is, it is a rhombus). In a final step we will also show that the polygon must be equiangular.

4. Overdetermined problem for triangles

In this section we will give two alternative proofs of Theorem 1.2 for triangles, i.e., in the case $N = 3$. The ideas used in both proofs will appear in the next section when we prove the theorem for quadrilaterals.

Remark 4.1. Notice that, in the proof of Theorem 1.2 for triangles, we will use only condition (1.6). In fact, (1.5) is satisfied by *every* triangle, since the operation of sliding one side and rescaling to restore the area leaves the triangle unchanged, and the corresponding first variation is equal to zero.

4.1. **Equilateral triangles via reflection arguments.** In the first proof we will use reflection arguments to obtain that if $\mathcal{P} \in \mathscr{P}_3$ satisfies (1.6), then $\mathcal{P}$ is equilateral.

First proof of Theorem 1.2 in the case $N = 3$. Let $\mathcal{P} \in \mathscr{P}_3$ be an arbitrary triangle and assume that $\mathcal{P}$ satisfies the condition (1.6). Without loss of generality, assume that $\mathcal{P}$ is translated and rotated so that the midpoint M_3 of the side $\overline{P_3P_1}$ coincides with the origin of the (x_1, x_2)-plane, and the side $\overline{P_1P_3}$ lies on the x_1-axis.

We show that, assuming condition (1.6) holds on the side $\overline{P_3P_1}$, we have $\theta_1 = \theta_3$. Suppose by contradiction $\theta_1 < \theta_3$. Let $\widetilde{\mathcal{P}}$ denote the reflection of $\mathcal{P}$ with respect to the x_2-axis, and define the sets $D \coloneqq \mathcal{P} \backslash \widetilde{\mathcal{P}}$ and $\widetilde{D} \coloneqq \widetilde{\mathcal{P}} \backslash \mathcal{P}$ (see Figure 4).

Let $x \in \overline{M_3P_3}$ and denote by $\widetilde{x} \in \overline{P_1M_3}$ the reflection of x in the x_2-axis. Then

$$\begin{aligned} v_{\widetilde{\mathcal{P}}}(x) - v_{\mathcal{P}}(x) &= \int_{\widetilde{\mathcal{P}}} K(|x-y|)\,\mathrm{d}y - \int_{\mathcal{P}} K(|x-y|)\,\mathrm{d}y \\ &= \int_{\widetilde{D}} K(|x-y|)\,\mathrm{d}y - \int_{D} K(|x-y|)\,\mathrm{d}y \\ &= \int_{D} \Big(K(|\widetilde{x}-y|) - K(|x-y|) \Big)\,\mathrm{d}y < 0, \end{aligned} \tag{4.1}$$

since $|\widetilde{x} - y| > |x - y|$ for all $y \in D$ and K is strictly decreasing. This implies that $v_{\widetilde{\mathcal{P}}}(x) < v_{\mathcal{P}}(x)$ for all $x \in \overline{M_3P_3}$. Multiplying both sides by $|x - M_3|$ and integrating, then, yields

$$\int_{\overline{M_3P_3}} v_{\mathcal{P}}(x)|x-M_3|\,\mathrm{d}\mathcal{H}^1(x) > \int_{\overline{M_3P_3}} v_{\widetilde{\mathcal{P}}}(x)|x-M_3|\,\mathrm{d}\mathcal{H}^1(x) = \int_{\overline{P_1M_3}} v_{\mathcal{P}}(x)|x-M_3|\,\mathrm{d}\mathcal{H}^1(x),$$

which contradicts the condition (1.6) on $\overline{P_3P_1}$. This implies that $\theta_1 = \theta_3$, i.e., $\mathcal{P}$ is isosceles.

Repeating the argument for another pair of angles, say θ_1 and θ_2, we obtain that $\theta_1 = \theta_2 = \theta_3$, i.e., $\mathcal{P}$ is equilateral. □

Notice that, by the previous proof, it is sufficient to assume that (1.6) holds just for two of the three sides of a triangle in order to deduce that it is equilateral.

4.2. **Equilateral triangles via first variation arguments.** Our second proof is inspired by the arguments in [7, Section 2.2.2] where the authors study the interaction energy $\mathcal{E}$ under continuous Steiner symmetrizations. Instead, we will use the the volume-preserving variations in Definition 3.8 and show that the first variation of $\mathcal{E}$ along these perturbations is strictly positive unless the triangle is isosceles.

The main idea of the proof is to express the first variation of the energy using slices of the triangle and to compute the derivative of the interaction between two slices. Let us start by fixing our notation for this subsection. As before, let $\mathcal{P} \in \mathscr{P}_3$ and fix the coordinate axes so that the midpoint M_3 of the side $\overline{P_3P_1}$ coincides with the origin of the (x_1, x_2)-plane, the side $\overline{P_1P_3}$ lies on the x_1-axis, and the point P_2 is in the upper half-plane. Assume $\theta_1 > \theta_3$. For $x_2 > 0$, let

$$\mathcal{P}_{x_2} \coloneqq \big\{ x_1 \in \mathbb{R} \colon (x_1, x_2) \in \mathcal{P} \big\}.$$

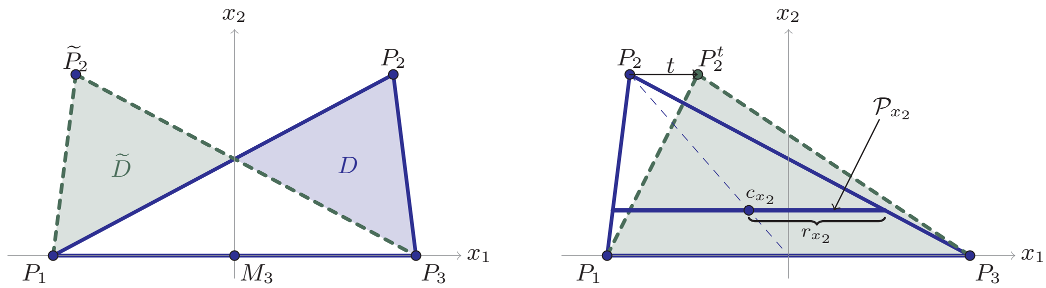

FIGURE 4. $\mathcal{P} \in \mathscr{P}_3$ translated so that the midpoint M_3 of the side $\overline{P_3P_1}$ coincides with the origin of the (x_1, x_2)-plane, and the side $\overline{P_1P_3}$ lies on the x_1-axis. The sets $D = \mathcal{P}\backslash\widetilde{\mathcal{P}}$ and $\widetilde{D} = \widetilde{\mathcal{P}}\backslash\mathcal{P}$ used in the reflection arguments (left) and area-preserving variations $\Phi_t(\mathcal{P})$ used in the first variation arguments (right).

Note that $\mathcal{P}_{x_2} \subset \mathbb{R}$ is an interval $(c_{x_2} - r_{x_2}, c_{x_2} + r_{x_2})$ with $r_{x_2} \geqslant 0$ and $c_{x_2} < 0$ since $\theta_1 > \theta_3$ (see Figure 4).

Let $\{\Phi_t\}_t$ denote the flow (3.13) introduced in the proof of Proposition 3.9. Then

$$\Phi_t(\mathcal{P}_{x_2}) = \mathcal{P}_{x_2} + \alpha x_2\, t \qquad \text{with} \qquad \alpha := \frac{1}{\ell_1 \sin\theta_1},$$

and

$$\Phi_t(\mathcal{P}) = \big\{(x_1, x_2) \colon x_1 \in \Phi_t(\mathcal{P}_{x_2})\big\}.$$

The next lemma shows that the derivative of the interaction between two slices is strictly positive for C^1 and even interaction kernels.

Lemma 4.2. *Let $W \in C^1(\mathbb{R})$ be an even function with $W'(r) < 0$ for $r > 0$. Then for every x_2, $y_2 > 0$ setting*

$$\mathcal{I}_W\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t) := \int_{\mathbb{R}}\int_{\mathbb{R}} W(x_1 - y_1)\chi_{\Phi_t(\mathcal{P}_{x_2})}(x_1)\chi_{\Phi_t(\mathcal{P}_{y_2})}(y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_W\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t) \geqslant C_W\,\alpha \min\{r_{x_2}, r_{y_2}\}|c_{x_2} - c_{y_2}|\,|x_2 - y_2|,$$

where

$$C_W = \min\big\{|W'(r)| \colon r \in \big[|c_{x_2} - c_{y_2}|/2, |c_{x_2} - c_{y_2}| + r_{x_2} + r_{y_2}\big]\big\}. \tag{4.2}$$

Proof. For simplicity of notation, we will drop the subscripts on the centers and radii of $\mathcal{P}_{x_2}$ and $\mathcal{P}_{y_2}$. Namely, let $\mathcal{P}_{x_2} = (c_x - r_x, c_x + r_x)$ and $\mathcal{P}_{y_2} = (c_y - r_y, c_y + r_y)$. Assume, without loss of generality, that $y_2 > x_2 > 0$. Then $r_y < r_x$ and $c_y < c_x < 0$, since $\theta_1 > \theta_3$.

Then

$$\mathcal{I}_W\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t) = \int_{-r_x}^{r_x}\int_{-r_y}^{r_y} W\big(x_1 - y_1 + c_x - c_y + \alpha x_2\, t - \alpha y_2\, t\big)\,\mathrm{d}y_1\,\mathrm{d}x_1,$$

and we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_W\big[\mathcal{P}_{x_2},\mathcal{P}_{y_2}\big](t) &= \alpha(x_2-y_2)\int_{-r_x}^{r_x}\int_{-r_y}^{r_y} W'\big((x_1-y_1)+(c_x-c_y)\big)\,\mathrm{d}y_1\,\mathrm{d}x_1\\ &= \alpha(x_2-y_2)\iint_R W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1,\end{aligned}$$

where R denotes the rectangle $\big[-r_x+(c_x-c_y), r_x+(c_x-c_y)\big]\times\big[-r_y, r_y\big]$ in the (x_1,y_1)-plane. Now, let

$$\begin{gathered}R^+ \coloneqq R\cap\{y_1>x_1\},\ R^- \coloneqq R\cap\{y_1<x_1\},\ \widetilde{R}^- \coloneqq R^-\cap\{x_1\leqslant r_x\},\\ \text{and } D\coloneqq\big[r_x+(c_x-c_y)/2, r_x+(c_x-c_y)\big]\times\big[-r_y, r_y\big],\end{gathered}$$

and note that R^+, $\widetilde{R}^-$, D are disjoint subsets of R, with $D\subset R^-$ (see Figure 5). Moreover $W'(x_1-y_1)>0$ on R^+ and $W'(x_1-y_1)<0$ on R^-. Since $x_2-y_2<0$ we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_W\big[\mathcal{P}_{x_2},\mathcal{P}_{y_2}\big](t) &= \alpha(x_2-y_2)\iint_R W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\ &\geqslant \alpha(x_2-y_2)\bigg[\iint_{R^+} W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &\quad+\iint_{\widetilde{R}^-} W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1+\iint_D W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\bigg].\end{aligned}\tag{4.3}$$

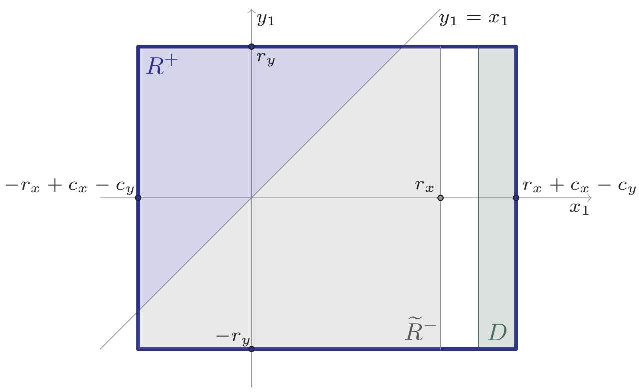

FIGURE 5. Subsets R^+, $\widetilde{R}^-$ and D of the rectangle R.

Since $R^+\cup\widetilde{R}^-$ is a rectangle with center $(\frac{c_x-c_y}{2},0)$, for every $h>0$ we have that $\mathcal{L}^1(R^+\cap\{y_1=x_1+h\})\leqslant\mathcal{L}^1(\widetilde{R}^-\cap\{y_1=x_1-h\})$. This, in turn, implies that

$$\iint_{R^+} W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1+\iint_{\widetilde{R}^-} W'(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\leqslant 0.$$

Returning to (4.3) and using the fact that W is even, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_W\big[\mathcal{P}_{x_2},\mathcal{P}_{y_2}\big](t) &\geqslant \alpha(y_2-x_2)\iint_D W'(y_1-x_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\ &\geqslant \alpha(y_2-x_2)|D|\min_{(x_1,y_1)\in D} W'(y_1-x_1)\\ &= \alpha(y_2-x_2)r_y(c_x-c_y)\min_{(x_1,y_1)\in D} W'(y_1-x_1)\\ &\geqslant C_W\,\alpha(y_2-x_2)r_y(c_x-c_y),\end{aligned}$$

which yields the result. □

Now we are ready to give the second proof of Theorem 1.2 in the case $N=3$.

Second proof of Theorem 1.2 in the case $N=3$. Let $\mathcal{P}\in\mathscr{P}_3$ be as above and assume by contradiction that $\theta_1>\theta_3$. As in the proof of Proposition 3.3, to avoid problems in differentiating we regularize the kernel by introducing a small parameter $\delta>0$ and obtain K_δ and $\mathcal{E}_\delta$ given by (3.2). Set $K_{\delta,l}(r):=K_\delta(\sqrt{l^2+r^2})=K(\sqrt{l^2+r^2}+\delta)$. Note that $K_{\delta,l}$ satisfies the assumptions of Lemma 4.2, namely it is a C^1, even function with $K'_{\delta,l}(r)<0$ for $r>0$. Then, by Fubini's theorem,

$$\mathcal{E}_\delta(\Phi_t(\mathcal{P}))=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K_\delta(|x-y|)\chi_{\Phi_t(\mathcal{P})}(x)\chi_{\Phi_t(\mathcal{P})}(y)\,\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2},\mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2$$

with $l:=|x_2-y_2|$. Hence, by Lemma 4.2, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}_\delta(\Phi_t(\mathcal{P})) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2},\mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2\\ &\geqslant \int_{\mathbb{R}}\int_{\mathbb{R}}C_{K_{\delta,l}}\,\alpha\min\{r_{x_2},r_{y_2}\}|c_{x_2}-c_{y_2}|\,|x_2-y_2|\,\mathrm{d}x_2\,\mathrm{d}y_2\geqslant C_\delta\end{aligned}$$

for some constant $C_\delta>0$, where $C_{K_{\delta,l}}$ is given by (4.2). Since C_δ is bounded away from zero uniformly in δ, as in the proof of Proposition 3.3 we can pass to the limit as $\delta\to0$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}(\Phi_t(\mathcal{P}))=\lim_{\delta\to0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}_\delta(\Phi_t(\mathcal{P}))>0.$$

However, due to Proposition 3.10, this contradicts the fact that $\mathcal{P}$ satisfies the condition (1.6). Therefore $\theta_1=\theta_3$, i.e., $\mathcal{P}$ is isosceles. Repeating this argument for all pairs of angles, we get that $\theta_1=\theta_2=\theta_3$; hence, $\mathcal{P}$ is equilateral. □

5. Overdetermined problem for quadrilaterals

In this section we prove Theorem 1.2 for quadrilaterals, i.e., in the case $N=4$. The proof exploits the same idea as in the triangle case, inspired by the arguments in [7], and uses a continuous symmetrization to prove that the stationarity conditions corresponding to sliding and tilting first enforce the quadrilateral to be equilateral; and then, via a reflection argument, they imply that the polygon is also equiangular.

Proof of Theorem 1.2 in the case $N=4$. Let $\mathcal{P}\in\mathscr{P}_4$ be an arbitrary quadrilateral satisfying the conditions (1.5) and (1.6) such that the diagonal between P_1 and P_3 lies on the x_1-axis, the midpoint of this diagonal coincides with the origin, and the vertex P_2 is in the upper half-plane. As in the case of a triangle, for any $x_2\in\mathbb{R}$, we let $\mathcal{P}_{x_2}=\{x_1\in\mathbb{R}\colon (x_1,x_2)\in\mathcal{P}\}\subset\mathbb{R}$. Then $\mathcal{P}_{x_2}=(c_{x_2}-r_{x_2},c_{x_2}+r_{x_2})$ for some $c_{x_2}\in\mathbb{R}$ and $r_{x_2}\geqslant0$ denoting the center and the radius of the slice $\mathcal{P}_{x_2}$, respectively. Also, we define $d_2 \coloneqq \operatorname{dist}\big(P_2, \{x_1 = 0\}\big)$ and $d_4 \coloneqq \operatorname{dist}\big(P_4, \{x_1 = 0\}\big)$.

We assume by contradiction that $\alpha_1 > \alpha_3$ where, as in the statement of Proposition 3.9, α_1 and α_3 are the angles between $\overline{P_1P_2}$ and $\overline{P_1P_3}$, and between $\overline{P_1P_3}$ and $\overline{P_2P_3}$, respectively (see Figure 6). Let $\{\Phi_t\}_t$ be the flow defined by

$$\Phi_t(x_1, x_2) \coloneqq \begin{cases} (x_1 + \beta^+ x_2\, t, x_2) & \text{if } x_2 > 0, \\ (x_1 - \beta^- x_2\, t, x_2) & \text{if } x_2 < 0, \end{cases} \tag{5.1}$$

where the constants $\beta^+, \beta^- \geqslant 0$ are to be chosen later, so that

$$\Phi_t(\mathcal{P}_{x_2}) = \begin{cases} \mathcal{P}_{x_2} + \beta^+ x_2\, t & \text{if } x_2 > 0, \\ \mathcal{P}_{x_2} - \beta^- x_2\, t & \text{if } x_2 < 0, \end{cases}$$

and $\Phi_t(\mathcal{P}) = \big\{(x_1, x_2) \colon x_1 \in \Phi_t(\mathcal{P}_{x_2})\big\}$.

Again, we regularize the kernel by introducing a small parameter $\delta > 0$ and take K_δ and $\mathcal{E}_\delta$ as in (3.2). Set $K_{\delta,l}(r) \coloneqq K_\delta(\sqrt{l^2 + r^2}) = K(\sqrt{l^2 + r^2} + \delta)$ and note that it satisfies the assumptions of Lemma 4.2. By Fubini's theorem, we write

$$\mathcal{E}_\delta(\Phi_t(\mathcal{P})) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_\delta(|x-y|)\chi_{\Phi_t(\mathcal{P})}(x)\chi_{\Phi_t(\mathcal{P})}(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2,$$

where

$$\begin{aligned} \mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t) &\coloneqq \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\delta,l}(x_1 - y_1)\chi_{\Phi_t(\mathcal{P}_{x_2})}(x_1)\chi_{\Phi_t(\mathcal{P}_{y_2})}(y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= \int_{-r_{x_2}}^{r_{x_2}}\int_{-r_{y_2}}^{r_{y_2}} K_{\delta,l}\big(x_1 - y_1 + c_{x_2} - c_{y_2} + \xi(x_2)t - \xi(y_2)t\big)\,\mathrm{d}y_1\,\mathrm{d}x_1 \end{aligned}$$

with $l = |x_2 - y_2|$ and

$$\xi(s) \coloneqq \begin{cases} \beta^+ s & \text{if } s > 0, \\ -\beta^- s & \text{if } s < 0. \end{cases} \tag{5.2}$$

Now, differentiating the energy yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}_\delta(\Phi_t(\mathcal{P})) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &= \int_0^{d_2}\int_0^{d_2} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &\quad + 2\int_0^{d_2}\int_{-d_4}^{0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &\quad + \int_{-d_4}^{0}\int_{-d_4}^{0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2. \end{aligned}$$

By using Lemma 4.2 to estimate the first integral (since $\beta^+ \geqslant 0$) and the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t) = \big(\xi(x_2) - \xi(y_2)\big)\int_{-r_{x_2}}^{r_{x_2}}\int_{-r_{y_2}}^{r_{y_2}} K'_{\delta,l}\big((x_1 - y_1) + (c_{x_2} - c_{y_2})\big)\,\mathrm{d}y_1\,\mathrm{d}x_1$$

to rewrite the second integral, we have that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}_\delta(\Phi_t(\mathcal{P}))\\
&\quad\geqslant \beta^+\int_0^{d_2}\int_0^{d_2} C_{K_{\delta,l}}\min\{r_{x_2},r_{y_2}\}|c_{x_2}-c_{y_2}|\,|x_2-y_2|\,\mathrm{d}x_2\,\mathrm{d}y_2\\
&\qquad+2\int_0^{d_2}\int_{-d_4}^{0}(\beta^+x_2+\beta^-y_2)\int_{-r_{x_2}}^{r_{x_2}}\int_{-r_{y_2}}^{r_{y_2}} K'_{\delta,l}\big((x_1-y_1)+(c_{x_2}-c_{y_2})\big)\,\mathrm{d}y_1\,\mathrm{d}x_1\,\mathrm{d}y_2\,\mathrm{d}x_2\\
&\qquad+\int_{-d_4}^{0}\int_{-d_4}^{0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2},\mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2
\end{aligned}
\tag{5.3}
$$

where $C_{K_{\delta,l}}$ is given by (4.2).

Now, let for $x_2\in[0,d_2]$ and $y_2\in[-d_4,0]$

$$
\mathcal{I}_\delta(x_2,y_2):=(\beta^+x_2+\beta^-y_2)\int_{-r_{x_2}}^{r_{x_2}}\int_{-r_{y_2}}^{r_{y_2}} K'_{\delta,l}\big((x_1-y_1)+(c_{x_2}-c_{y_2})\big)\,\mathrm{d}y_1\,\mathrm{d}x_1.
$$

We will show that $\mathcal{I}_\delta(x_2,y_2)\geqslant 0$ for some β^+, $\beta^-\geqslant 0$. In order to achieve this estimate we distinguish between two cases: (i) P_4 lies in the fourth quadrant of the (x_1,x_2)-plane, or (ii) P_4 lies in the third quadrant of the (x_1,x_2)-plane (see Figure 6).

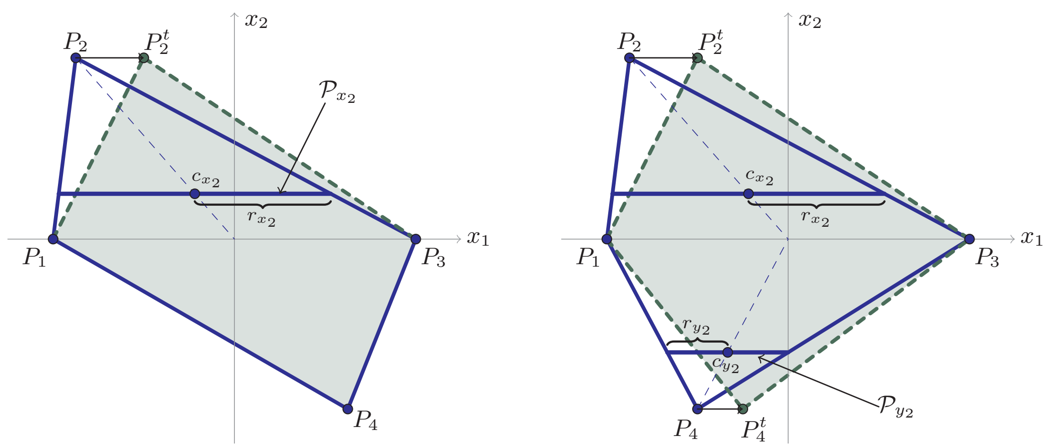

FIGURE 6. The variation considered in the proof of Theorem 1.2, $N=4$: Case (i) (left) and Case (ii) (right).

Case (i). (P_4 lies in the fourth quadrant of the (x_1,x_2)-plane.) Since, by assumption, $\alpha_1>\alpha_3$, for $x_2>0$ the center of the slice $\mathcal{P}_{x_2}$ is given by $c_{x_2}=\zeta x_2$ where $\zeta<0$ is the slope of the line passing through the origin and the vertex P_2. We choose $\beta^+=-\zeta>0$ and $\beta^-=0$ in (5.1). Note that, since P_2 and P_4 are on the opposite sides of the x_2-axis, we have that $c_{x_2}-c_{y_2}<0$.

As in the proof of Lemma 4.2, we have

$$
\mathcal{I}_\delta(x_2,y_2)=\beta^+x_2\iint_R K'_{\delta,l}(x_1-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1,
$$

where

$$
R:=\big[-r_{x_2}+c_{x_2}-c_{y_2},r_{x_2}+c_{x_2}-c_{y_2}\big]\times\big[-r_{y_2},r_{y_2}\big].
\tag{5.4}
$$

Since R is a rectangle centered at $(\frac{c_{x_2}-c_{y_2}}{2}, 0)$ and $c_{x_2} - c_{y_2} < 0$, we have that for every $h > 0$

$$\mathcal{L}^1\big(R \cap \{y_1 = x_1 + h\}\big) \geqslant \mathcal{L}^1\big(R \cap \{y_1 = x_1 - h\}\big). \tag{5.5}$$

Therefore

$$\begin{aligned}
\mathcal{I}_\delta(x_2, y_2) &= \frac{\beta^+ x_2}{\sqrt{2}} \int_0^{+\infty} \left(\int_{R\cap\{y_1=x_1+h\}} K'_{\delta,l}(-h)\,\mathrm{d}\mathcal{L}^1 + \int_{R\cap\{y_1=x_1-h\}} K'_{\delta,l}(h)\,\mathrm{d}\mathcal{L}^1 \right) \mathrm{d}h \\
&= \frac{\beta^+ x_2}{\sqrt{2}} \int_0^{+\infty} K'_{\delta,l}(h) \Big(\mathcal{L}^1(R\cap\{y_1 = x_1 - h\}) - \mathcal{L}^1(R\cap\{y_1 = x_1 + h\}) \Big)\,\mathrm{d}h \\
&\geqslant 0,
\end{aligned}$$

by (5.5) and the fact that $K'_{\delta,l}(h) < 0$ for $h > 0$.

Case (ii). (P_4 lies in the third quadrant of the (x_1, x_2)-plane.) Now, given any $x_2 \in \mathbb{R}$, the center of the slice $\mathcal{P}_{x_2}$ is given by $c_{x_2} = \zeta x_2$ if $x_2 > 0$ and by $c_{x_2} = \eta x_2$ if $x_2 < 0$, where $\zeta < 0$ and $\eta > 0$ are two constants given by the slopes of the lines passing through the origin and the vertices P_2 and P_4, respectively. In this case we choose $\beta^+ = -\zeta > 0$ and $\beta^- = \eta > 0$ in (5.1), and, as before, rewrite $\mathcal{I}_\delta(x_2, y_2)$ as

$$\mathcal{I}_\delta(x_2, y_2) = (\beta^+ x_2 + \beta^- y_2) \iint_R K'_{\delta,l}(x_1 - y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1,$$

where R is the rectangle defined by (5.4).

Suppose $\beta^+ x_2 + \beta^- y_2 > 0$. Then

$$c_{x_2} - c_{y_2} = \zeta x_2 - \eta y_2 = -(\beta^+ x_2 + \beta^- y_2) < 0,$$

and, as in the previous case, R is a rectangle centered on the negative x_1-axis, hence we get that $\mathcal{I}_\delta(x_2, y_2) \geqslant 0$.

Suppose $\beta^+ x_2 + \beta^- y_2 < 0$. Then $c_{x_2} - c_{y_2} > 0$, and therefore the center of the rectangle R is on the positive x_1-axis: it follows that for every $h > 0$

$$\mathcal{L}^1\big(R \cap \{y_1 = x_1 + h\}\big) \leqslant \mathcal{L}^1\big(R \cap \{y_1 = x_1 - h\}\big).$$

Hence we get that

$$\int_0^{+\infty} K'_{\delta,l}(h) \Big(\mathcal{L}^1(R\cap\{y_1 = x_1 - h\}) - \mathcal{L}^1(R\cap\{y_1 = x_1 + h\}) \Big)\,\mathrm{d}h \leqslant 0,$$

and since $\beta^+ x_2 + \beta^- y_2 < 0$, we conclude that $\mathcal{I}_\delta(x_2, y_2) \geqslant 0$.

Conclusion. We proved that in both cases, for a suitable choice of $\beta^+ > 0$ and $\beta^- \geqslant 0$, we have $\mathcal{I}_\delta(x_2, y_2) \geqslant 0$ for every $x_2 \in [0, d_2]$ and $y_2 \in [-d_4, 0]$.

Going back to (5.3) we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{E}_\delta(\Phi_t(\mathcal{P})) \geqslant \beta^+ \int_0^{d_2}\int_0^{d_2} & C_{K_{\delta,l}} \min\{r_{x_2}, r_{y_2}\} |c_{x_2} - c_{y_2}|\,|x_2 - y_2|\,\mathrm{d}x_2\,\mathrm{d}y_2 \\
&+ \int_{-d_4}^0 \int_{-d_4}^0 \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{I}_{K_{\delta,l}}\big[\mathcal{P}_{x_2}, \mathcal{P}_{y_2}\big](t)\,\mathrm{d}x_2\,\mathrm{d}y_2.
\end{aligned}$$

Concerning the second integral above, it is sufficient to observe that it is equal to zero in Case (i) (since $\beta^- = 0$), and it is nonnegative in Case (ii) as a consequence of Lemma 4.2

(since $\beta^- \geqslant 0$). Therefore, recalling the definition (4.2) of $C_{K_{\delta,l}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}_\delta(\Phi_t(\mathcal{P})) \geqslant C_\delta > 0,$$

for a constant C_δ bounded away from zero uniformly in δ. Again, as in Proposition 3.3, we can pass to the limit $\delta \to 0$ and get that $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{E}(\Phi_t(\mathcal{P})) > 0$. However, this contradicts, via Proposition 3.10, the fact that $\mathcal{P}$ satisfies (1.5) and (1.6). Therefore $\alpha_1 = \alpha_3$.

By swapping P_2 and P_4 in the arguments above yields that, in fact, $\theta_1 = \theta_3$. Now, repeating the same arguments for the vertices P_1 and P_3 (that is, taking the diagonal $\overline{P_2P_4}$ as the direction of symmetrization), we obtain that $\theta_2 = \theta_4$, i.e., that $\mathcal{P}$ is a rhombus.

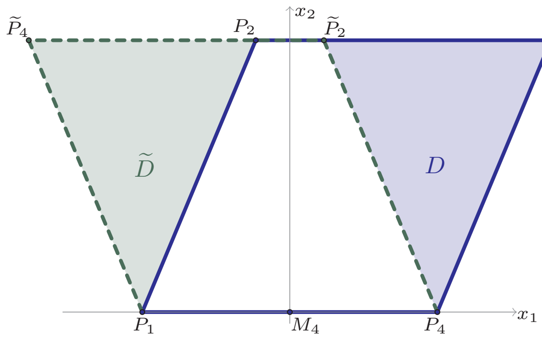

FIGURE 7. A reflection argument shows that if $\mathcal{P}$ is rhombus and satisfies (1.6) then $\mathcal{P}$ has to be a square. Here the reflection of $\mathcal{P}$ in the x_2-axis is the rhombus $\widetilde{\mathcal{P}}$ depicted with the dashed lines.

Finally, we are going to use a reflection argument similar to the one in the first proof of Theorem 1.2 in the case $N = 3$ in order to conclude that $\mathcal{P}$ is a square. Since $\mathcal{E}$ is invariant under rigid transformations, suppose the side $\overline{P_1P_4}$ lies on the x_1-axis and the midpoint M_4 coincides with the origin. Suppose $\theta_1 < \theta_4$. Let $\widetilde{\mathcal{P}}$ denote the reflection of $\mathcal{P}$ with respect to the x_2-axis, and define the sets $D \coloneqq \mathcal{P}\backslash\widetilde{\mathcal{P}}$ and $\widetilde{D} \coloneqq \widetilde{\mathcal{P}}\backslash\mathcal{P}$ (see Figure 7). Let $x \in \overline{M_4P_4}$ and denote by $\widetilde{x} \in \overline{P_1M_4}$ the reflection of x in the x_2-axis. Then the same calculation as in (4.1) shows that $v_{\widetilde{\mathcal{P}}}(x) - v_{\mathcal{P}}(x) < 0$. Again, multiplying both sides by $|x - M_4|$ and integrating, then, yields a contradicts with the condition (1.6); hence, $\theta_1 = \theta_4$, and we conclude that $\mathcal{P}$ is a square. $\square$

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